CMPS 242: HW3

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1. Linear Regression

   (a) MLE estimation of w, β. Here we maximize the log likelihood.

$$
\begin{aligned}
p(t|x,w,\beta) &= N(t|y(x,w),\beta^{-1}) \\
w_{MLE}, \beta_{MLE} &= \underset{w,\beta}{\text{argmax}}\ log \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\beta^{-1}}} exp\left(- \frac{(t_i - y(x_i,w))^2}{2\beta^{-1}} \right) \\
&= \underset{w,\beta}{\text{argmin}}\ \ - n\ log(\beta) + \beta \sum_{i=1}^{n} (t_i - y(x_i,w))^2
\end{aligned}
$$

Taking derivatives with respect to w_0, w_1, w_2, w_3, and setting equal to zero, we note that β falls out of the equation in each case, giving us 4 equations and four unknowns. This means that our MLE estimates for w won't depend on the variance β^{-1} and they will take the standard form,

$$
\begin{aligned}
X &= [1, x, x^2, x^3] \\
w &= (X^T X)^{-1} X^T t
\end{aligned}
$$

Further, taking the derivative of the log likelihood with respect to β yields the following.

$$
\begin{aligned}
\frac{\partial}{\partial \beta} \left(\underset{w,\beta}{\text{argmin}}\ \ - n\ log(\beta) + \beta \sum_{i=1}^{n} (t_i - y(x_i,w))^2 \right) &= 0 \\
\implies \frac{n}{\beta_{MLE}} &= \sum_{i=1}^{n} (t_i - y(x_i,w))^2 \\
\implies \beta_{MLE}^{-1} &= \frac{1}{n} \sum_{i=1}^{n} (t_i - y(x_i,w_{MLE}))^2
\end{aligned}
$$

Using this approach yields the following estimates for w, β for 100, 1000, 10000 test points. We note that w_1, w_2, w_3 remain relatively constant while the intercept seems to get closer to the true value of .2. We also note that our estimate for the variance seems to improve once we get more than 100 training examples. Interestingly, if we use the optimize module in scipy with method BFGS to perform

n training points	w_0	w_1	w_2	w_3	β^{-1}
100	0.2401381	2.0000394	0.9999990	2.9999995	0.8202118
1000	0.1834306	1.9981898	0.9999978	3.0000003	1.002905
10000	0.1903286	2.0004489	0.9999998	2.9999999	1.014826

this same task, optimizing over β and w simultaneously, we get very similar results for w, but β is computed to be around 3.35 in each case. I'm not certain why this would be. Also we note that accurate convergence of our minimization is very dependent on having a close starting value.

(b) Repeat part a but with a fifth degree polynomial rather than a 3rd degree polynomial. This yields the following table of results using the mathematical approach above. We see that the MLE correctly sets w_4, w_5 to be very nearly 0.

n training points	w_0	w_1	w_2	w_3	w_4	w_5	β^{-1}
100	.2224	2.0089	1.0000	2.99999	-7.5716e-09	3.9693e-10	0.7998198
1000	0.14827	1.9964	1.0000	3.0000	-4.4164e-09	-7.401052e-11	1.000897
10000	.19577	2.0012	.99999	3.0000	6.5339e-10	3.3887e-11	1.014

This time with the scipy approach we see similar but more extreme results. In particular, the coefficient estimates that we converge to seem even more dependent on the point we initialize the solver at. Often it will set w_4, w_5 to be small, but our estimates for w_0, w_1, w_2, w_3 can vary dramatically.

(c) Bayesian Linear Regression
In this problem we seek to maximize the following equation with respect to w and alpha, where $\beta = \alpha = 1$.

$$ln\ p(w|t) = -\frac{\beta}{2}\sum_{i=1}^{n}\left(t_i - w^T\phi(x_i)\right)^2 - \frac{\alpha}{2}w^Tw + const.$$

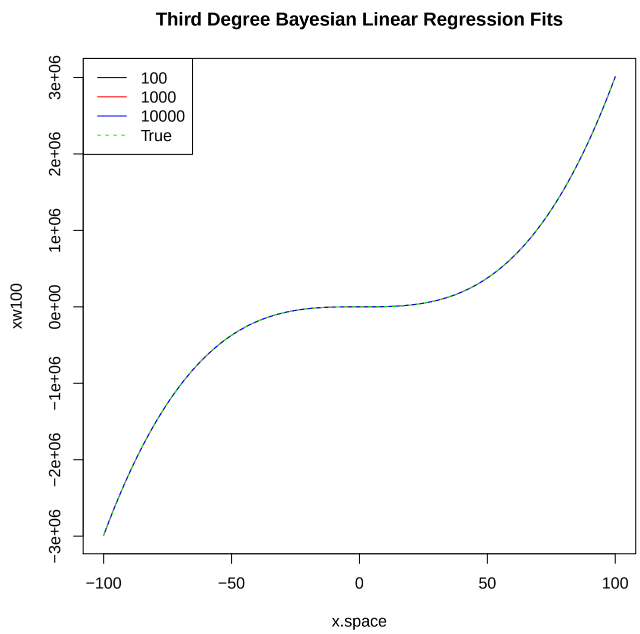
From the book we note that the posterior distribution is as follows.

$$\begin{aligned} p(w|t) &= N(w|m_N, S_N) \\ m_N &= \beta S_N \Phi^T t \\ S_N^{-1} &= \alpha I + \beta\Phi^t\Phi \\ \implies m_N &= (I + \Phi^t\Phi)^{-1}\Phi^T t \end{aligned}$$

This equation will give us the MLE's for w that we need. The results are summarized in the following tables below. Some plots of the estimated curves are also provided.

Training Examples	w_0	w_1	w_2	w_3
100	0.2345511	1.9999760	0.9999999	2.9999995
1000	0.1830405	1.9981858	0.9999979	3.0000003
10000	0.1902861	2.0004486	0.9999999	2.9999999
True	0.2	2	1	3

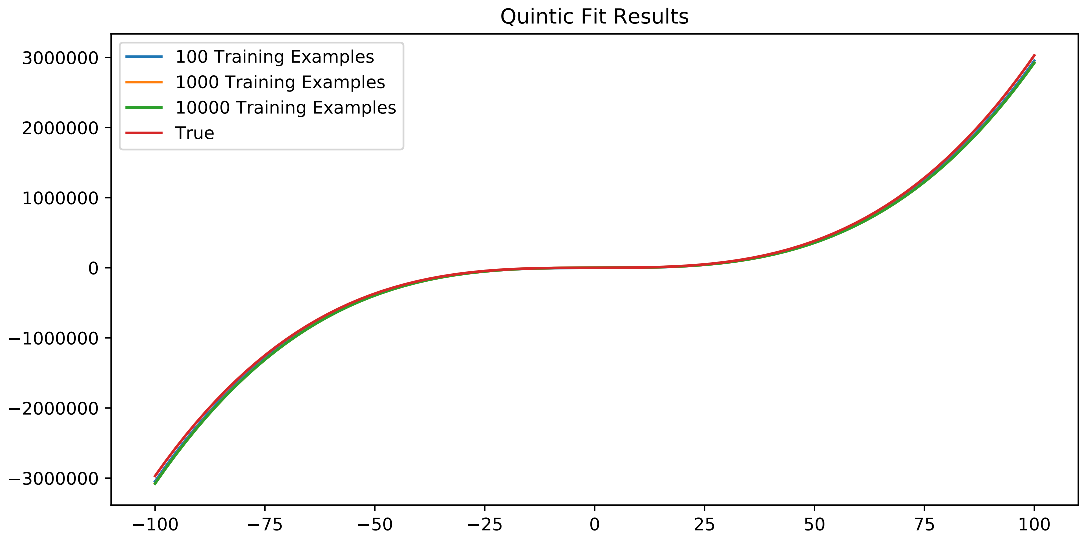
Table 1: Cubic Polynomial Fit Coefficient Results



As we can see looking at the table of coefficients and at the plot of Third Degree Bayesian Regression fits, the bayesian regression approach gives very accurate results, with all curves lining up almost identically. The fifth degree polynomial fits worked fairly well also. Unfortunately however, I was unable to invert the matrix $(I + \Phi^T \Phi)$ and so a numerical minimization approach had to be used. The results from this approach are still graphically good, though we note in the table that the coefficient estimates are much less exact than with the third degree bayesian regression.

Training Examples	w_0	w_1	w_2	w_3	w_4	w_5
100	-2033.495	-5.526	2.00095	3.004	-9.549e-5	-3.5633e-7
1000	80.921	-8.63	.9738	3.00479	2.156e-6	-4.219e-7
10000	-6.157	-8.849	1.004	3.005	-4.489e-7	-4.5357e-7
True	0.2	2	1	3	0	0

Table 2: Quintic Polynomial Fit Coefficient Results



2. Naive Bayes Text Classification

   In this problem, we perform text classification with a Naive Bayes Classifier. We're given 12 datasets, Enron1.ham, Enron2.ham, ... Enron6.ham, Enron1.spam, Enron2.spam, ..., Enron6.spam. We use Enron1 through Enron5 to train a classifier to differentiate between ham and spam, and test on Enron6. We classify as ham if:

$$
\begin{aligned}
& P(ham|data) > P(spam|data) \\
\implies & \frac{P(data|ham)P(ham)}{P(data)} > \frac{P(data|spam)P(spam)}{P(data)} \\
\implies & P(data|ham)P(ham) > P(data|spam)P(spam) \\
& P(ham) = \frac{15045}{15045 + 12669} \\
& P(spam) = 1 - P(ham) \\
& P(data|ham) = \prod_{n=1}^{N} P(\text{test example word n}|ham) \\
& P(data|spam) = \prod_{n=1}^{N} P(\text{test example word n}|spam)
\end{aligned}
$$

   In the above equations N indicates the number of words in a given test email. Different prior probabilities arising from different numbers of spam and ham emails in the training set are accounted for in the above equations. For computational reasons, all calculations are done on the log scale. Further, all computations are done without the use of python packages like sklearn. Below we report accuracy results for versions with and without laplace smoothing. Accuracy is measured as the proportion of emails in the Enron6 dataset that are classified correctly.

   As we can see in the table above, adding laplace smoothing dramatically improves accuracy. This can in large part be explained by a large issue that arises when laplace

Smoothing	Ham Accuracy	Spam Accuracy	Total Accuracy
None	973/1500	3274/4499	$4247/5999 \simeq 0.708$
Laplace	1431/1500	4465/4499	$5896/5999 \simeq 0.983$

Table 3: Naive Bayes Classifier Accuracy Results

smoothing isn't included. Namely, if laplace smoothing isn't implemented, we will inevitably encounter words in the test data that aren't included in the training data. In this situation the Niave Bayes classifier without laplace smoothing will set the probability of that particular word given a certain class to zero. After a log transformation, we're left with a $P(class|data) = -\infty$. If both class's training sets don't include a word in the test email, then we're left with $P(class\ 0|data) = P(class\ 1|data) = -\infty$. In this situation, I assigned that email to the ham class with probability equal to the ham prior. A more sophisticated approach involving ignoring the word not existent in the training set would probably perform better. We also chose to not do anything special with punctuation. It would be interesting to see if removing punctuation improved the performance of the classifier. Below we include a list of the most discriminative words based on the learned probabilities. In order to get this table, we compute the log probability associated with each word for spam and ham, and then find the absolute value of the difference between these two log probabilities. Sorting on this difference gives us table four.

Word	abs(log probability difference)
enron	10.411474
kaminski	7.968558
dynegy	7.958526
pills	7.898633
viagra	7.810160
ect	7.561901
computron	7.483230

Table 4: Most Discriminative Words